Bull. Faculty of Liberal Arts, Nagasaki Univ., (Natural Science), 28(1), 1–9 (July, 1987)

Auslander-Reiten quivers of orders

Dedicated to Professor Hisao Tominaga on his 60th birthday

Kenji NISHIDA

(Received March 16, 1987)

1. Introduction

Let R be a complete discrete valuation ring with prime element π and residue field $\boldsymbol{k}$. Let Λ be an R-order in the semisimple finite dimensional algebra over the quotient field of R and Γ a hereditary R-order in the same algebra such that $J=\operatorname{rad}\Gamma\subset\Lambda\subset\Gamma$. Put $A=\Lambda/J$ and $B=\Gamma/J$. Then A is a $\boldsymbol{k}$-subalgebra of the semisimple $\boldsymbol{k}$-algebra B. We assume that Λ is basic, so that A is, too. By the technical reason we assume that Λ is not a hereditary order whose structure is completely determined (see, for example [4]). In the previous paper [3], we showed that latt Λ, the category of all right Λ-lattices, is representation equivalrnt to $\mathrm{mod}_{\mathrm{sp}}{}^{1}$ C with $C=\begin{pmatrix} A & DI \\ 0 & K \end{pmatrix}$ under the above situation. But, as for the Auslander-Reiten quiver of latt Λ, we didn't give any information there, while Roggenkamp[8] gives the method of constructing it under the further assumption that rad Λ decomposes into a projective Λ-lattice and a Γ-lattice and Λ is of finite lattice type. In fact, for $C'=\begin{pmatrix} A & B \\ 0 & B \end{pmatrix}$, the relation of irreducible maps between latt Λ and the subcategory $\mathscr{C}$ of mod C' is determined in [8]. It is noted that C' is a hereditary algebra under the assumption of [8]. When Λ is of finite lattice type, the Auslander-Reiten quiver of latt Λ is obtained from that of $\mathscr{C}$ by identifying some vertices of $\mathscr{C}$[8, III Theorem V].

In this paper, we generalize the results of [8] and study the problem how the Auslander-Reiten quiver of latt Λ is constructed from that of $\mathrm{mod}_{\mathrm{sp}}$ C under our situation. In order to describe our results more precisely we prepare notation. We identify a C'-module with a triple (X, Y, ϕ) where X is a right A-module, Y a right B-module and $\phi : X\otimes_A B\rightarrow Y$ a B-homomorphism. Let C be a full subcategory of mod C' consisting of the modules of the form (X, Y, ϕ) such that X is finitely generated, the adjoint $\tilde{\phi}\in\mathrm{Hom}_A$ (X, Y) of ϕ is injective and $(\mathrm{Im}\,\phi)B=Y$. For a Λ-lattice M, put $\overline{M}=M/MJ$,

$\overline{M\Gamma} = M\Gamma/MJ$ and $\bar{\phi}: \overline{M} \to \overline{M\Gamma}$ the canonical inclusion. Since $\bar{\phi}$ induces ϕ': $\overline{M}\otimes_A B \to \overline{M\Gamma}$, we may put $H(M) = (\overline{M}, \overline{M\Gamma}, \phi')$. We sometimes omit ϕ' and write $H(M) = (\overline{M}, \overline{M\Gamma})$. Then H is a functor from latt Λ to mod C' and the following was proved in [2, 5].

THEOREM A. *There exists a representation equivalence* latt $\Lambda \approx \mathscr{C}$ *induced from H.*

A $\boldsymbol{k}$-algebra Z is called a right peak ring, if the socle soc Z of Z as a right Z-module is projective. A right peak ring is introduced and studied in [9] when soc Z is homogeneous, however, almost all results about a right peak ring with a homogeneous socle hold for nonhomogeneous case. We next define a right peak $\boldsymbol{k}$-algebra which plays a valuable part in this paper. Let $G_1, \ldots, G_t$ be the representatives of nonisomorphic indecomposable projective right Γ-lattices. Put $S_i = G_i/G_iJ (i=1, \ldots, t)$ and $I = S_1 \oplus \ldots \oplus S_t$. Then $S_1, \ldots, S_t$ are the representatives of nonisomorphic simple right B-modules. For $K_i = \mathrm{End}_B\, S_i (i=1, \ldots, t)$ put $K = K_1 \otimes \ldots \otimes K_t$ a product of division rings. Let D be the usual duality $D(-) = \mathrm{Hom}_k(-, \boldsymbol{k})$. Then it is easily seen that a ring $C = \begin{pmatrix} A & DI \\ 0 & K \end{pmatrix}$ is a right peak ring by [9]. For $u = (X, Y, \psi) \in \mathrm{mod}\ C'$, put $H'(u) = (X, Y\otimes_B DI, \bar{\psi})$, where $\bar{\psi}: X\otimes_A DI \to Y\otimes_B DI$ is given by $\bar{\psi}(x\otimes g) = \psi(x\otimes 1)\otimes g$ $(x \in X, g \in DI)$. Then H' is a functor from mod C' to mod C which is a category equivalence. Let $\mathrm{mod}_{\mathrm{sp}}\, C$ be the full subcategory of mod C whose modules have projective socles and $\mathrm{mod}_{\mathrm{sp}}^1\, C$ the full subcategory of $\mathrm{mod}_{\mathrm{sp}}\, C$ consisting of all modules having no direct summand of the form $(0, K_i, 0) (i=1, \ldots, t)$, that is, having no simple projective direct summand. Then the following was proved in [3, Theorem 4].

THEOREM B. *There exists a representation equivalence* latt $\Lambda \approx \mathrm{mod}_{\mathrm{sp}}^1\, C$ *induced from* $\Phi = H'H$.

Since $\mathrm{mod}_{\mathrm{sp}}\, C$ has been extensively studied by Simson[9] when C is a right peak ring, the well understanding of the functor Φ contributes the investigation of the representation theory of the orders. We consider the behavior of irreducible maps, almost split sequences under the functor Φ and then provide the method of constructing the Auslander-Reiten quiver of latt Λ from that of $\mathrm{mod}_{\mathrm{sp}}\, C$. Hence our main theorem is the following, where the undefined notation will be explained in the next section.

THEOREM C. *The Auslander-Reiten quiver of* latt Λ *is obtained from that of* $\mathrm{mod}_{\mathrm{sp}}\, C$ *by identifying the indecomposable injective C-module* $\Phi(G)$ *with the simple projective C-module* $(0, \mathrm{End}_B\, \overline{\theta(G)}, 0)$ *for every indecomposable projective* Γ*-lattice G.*

We assume throughout the paper that all modules are finitely generated. The reader is referred to [1, 8] for the definition and the properties of irreducible maps, almost split sequences, Auslander-Reiten quivers and to [9] for properties of $\mathrm{mod}_{\mathrm{sp}}\, C$ for a right peak ring C, for example, it was proved in [9] that $\mathrm{mod}_{\mathrm{sp}}\, C$ has almost split sequences, enough injectives, and so on.

2. Auslander-Reiten quivers of latt Λ

The functor Φ is essentially dominated by the functor H. The useful results about H is provided in [5, §1], so we will freely use them. Firstly we begin with a lemma which seems to be well-known, but we provide the proof here for the completeness.

LEMMA 1. *For* M, $M' \in$ latt Λ, $f: M \to M'$ *is a splitting monomorphism (respectively epimorphism) if and only if* $\Phi(f) : \Phi(M) \to \Phi(M')$ *is a splitting monomorphism(respectively epimorphism).*

PROOF. Since H' is a category equivalence, we prove the lemma for H. Let $H(f)$ be a splitting monomorphism. Then there exists $g: M' \to M$ such that $H(g)H(f) = 1_{H(M)}$. Define $f_1 : M\Gamma \to M'\Gamma$ and $g_1 : M'\Gamma \to M\Gamma$ by $f_1(m\gamma) = f(m)\gamma$ and $g_1(m'\gamma) = g(m')\gamma$ for $m \in M$, $m' \in M'$, $\gamma \in \Gamma$. By assumption it holds that $g_1 f_1(M\Gamma) + MJ = M$, so that $g_1 f_1(M\Gamma) = M$ by Nakayama's Lemma. Since Ker $g_1 f_1 = 0$ by the rank arguement, $g_1 f_1$ is an isomorphism. On M $g_1 f_1$ equals gf, thus gf is also an isomorphism. Therefore, f is a splitting monomorphism. The rest of the proof is almost trivial.

The following two lemmas are also sbtained in [8, Proposition 4. 4] with a slightly different manner.

LEMMA 2. *Let* M, $N \in$ latt Λ *with M indecomposable and let* $f: M \to N$ *be an irreducible map. Then* $\Phi(f) = 0$ *if and only if* $f(M)$ *is a direct summand of NJ. Moreover, in this case N is a projective* Λ*-lattice.*

PROOF. It holds that $\Phi(f) = 0 \Leftrightarrow H(f) = 0 \Leftrightarrow f(M) \subset NJ$. Thus if $\Phi(f) = 0$, then f factors through $M \xrightarrow{f} NJ \subset N$. Since f is irreducible, $f: M \to NJ$ is a splitting monomorphism. The converse is obvious. The rest of the statement is showed in [8, Proposition 4. 4].

LEMMA 3. *Let M, N be as in Lemma 2. Assume that $\Phi(f)$ is a nonzero nonisomorphism for $f: M \to N$. Then f is irreducible in* latt Λ *if and only if $\Phi(f)$ is irreducible in* $\mathrm{mod}_{\mathrm{sp}}$ C.

PROOF. Let $f_1: M\Gamma \to N\Gamma$ be as in the proof of Lemma 1, and let $\bar{f}: \bar{M} \to \bar{N}$, resp. $\bar{f_1}: \overline{M\Gamma} \to \overline{N\Gamma}$ be induced from f, resp. f_1. Assume that $\Phi(f)$ factors through $\Phi(M) \xrightarrow{g} u \xrightarrow{h} \Phi(N)$. Since only simple projective C-modules are not in Im Φ among the modules in $\mathrm{mod}_{\mathrm{sp}}$ C by Theorem B, we can assume that $u \in \mathrm{Im}\ \Phi$, so that $u=\Phi(X)$, $g=\Phi(\alpha)$, $h=\Phi(\beta)$ for $X \in$ latt Λ, $\alpha: M \to X$, $\beta: X \to N$. Put $\phi = f - \beta\alpha: M \to N$. Then Im $\phi \subset NJ$ by $\bar{f} = \overline{\beta\alpha}$. Hence $f = \phi + \beta\alpha = (\iota, \beta)(\phi, \alpha)$, where $\iota: NJ \to N$ is a canonical inclusion. If f is irreducible, then either (ι, β) is a splitting epimorphism or (ϕ, α) is a splitting monomorphism. Assume that (ι, β) is a splitting epimorphism. Then there exists $(\gamma, \beta'): N \to NJ \oplus X$ such that $\beta\beta' + \iota\gamma = 1_N$. It holds that Im $\beta\beta' + NJ = N$ and so Im $\beta\beta' = N$. Therefore, $\beta\beta': N \to N$ is an isomorphism. Hence we have that β is a splitting epimorphism, and $H(\beta)$ is, too. Similarly, if (Φ, α) is a splitting monomorphism, then $H(\alpha)$ is a splitting monomorphism. Hence we showed that $\Phi(f)$ is irreducible. The converse is obvious by Lemma 1.

PROPOSITION 1. *For every Γ-lattice M, $\Phi(M)_C$ is injective.*

PROOF. We assume that $M = G_i$ for some $i (1 \leqq i \leqq t)$. Thus $H(G_i) = (S_i, S_i, \phi_i)$ with $\phi_i: S_i \otimes_A B \to S_i$ canonically, and $\Phi(G_i) = (S_i, S_i \otimes_B DI, \psi_i)$. Since $S_i \otimes_B DI \cong K_i$ and $S_i \otimes_A DI \cong S_i \otimes_A I^*$ with $I^* = \mathrm{Hom}_K(I, K)$ we conclude that the K_i-homomorphism $\psi_i: S_i \otimes_A DI \to K_i$ is the canonical evaluation map and $\Phi(G_i) \cong (S_i, K_i, \psi_i)$. Then $D\Phi(G_i) \cong ({}_A S_i^*, {}_K K_i, \lambda_i)$, where $\lambda_i: I^* \otimes_K K_i \to S_i^*$ is given by $[(f_j) \otimes y \to f_i y]$ for $f_j \in S_j^*$ $(j = 1, \dots, t)$, is a projective left C-module by the form of C. Therefore, $\Phi(G_i)_C$ is injective.

COROLLARY 1. *Let $f: M \to N$ be an irreducible map between indecomposable Λ-lattices such that M is a Γ-lattice and $\Phi(f) \neq 0$. Then M and N are isomorphic Λ-lattices.*

PROOF. If $\Phi(f)$ is not an isomorphism, then $\Phi(f)$ is irreducible by Lemma 3. Since $\Phi(M)$ is injective by Proposition 1, $\Phi(f)$ is an epimorphism. However, $\Phi(M)$ has a projective simple socle and $\Phi(N) \in \mathrm{mod}_{\mathrm{sp}}$ C, a contradition. Thus $\Phi(f)$ is an isomorphism and $M \cong N$.

Since the Auslander-Reiten quiver of an order such that there exists an irreducible map $M \to M$ is completely determined by [11] (cf. also [7, Theorem 3. 11]), we assume in the following that there exists no irreducible map $M \to M$ for every Λ-lattice M.

Corollary 2. *Let $f: M \to N$ be an irreducible map between Λ-lattices. Then $\Phi(f) \neq 0$ if and only if M is not a Γ-lattice.*

The next proposition is essential, because it assures that the almost split sequences, except those starting from a Γ-lattice, are preserved by Φ.

Proposition 2. *Let $0 \to M \xrightarrow{f} M' \xrightarrow{g} M'' \to 0$ be an exact sequence of non-zero Λ-lattices such that f is irreducible and M is indecomposable and not a Γ-lattice. Then $0 \to \Phi(M) \xrightarrow{\Phi(f)} \Phi(M') \xrightarrow{\Phi(g)} \Phi(M'') \to 0$ is exact.*

Proof. Let f_1 and g_1 be the same as in the proof of Lemma 1. Then we have the following two exact sequences,

$$0 \to \ker g_1 \to M'\Gamma \xrightarrow{g_1} M''\Gamma \to 0,$$
$$0 \to M\Gamma \xrightarrow{f_1} \ker g_1 \to T \to 0,$$

where T is a torsion R-module. Since $H(f) \neq 0$ by Corollary 2, it holds that $\mathrm{Im}\, f_1 \not\subset (\ker g_1)J$. Thus we have a decomposition $\ker g_1 = X_1 \oplus X_2$, where X_1, X_2 are Γ-lattices with $X_1 \neq 0$ such that $f_1 = (f'_1, f'_2)$ for $f'_i: M\Gamma \to X_i\,(i=1, 2)$, $\mathrm{Im}\, f'_i = X_1$ and $\mathrm{Im}\, f'_2 \subset X_2 J$. Considering the induced map $\overline{f_i} = (\overline{f'_1}, \overline{f'_2}):$ $\overline{M\Gamma} \to \overline{X_1} \oplus \overline{X_2} = \overline{\ker g_1}$, we have $\overline{f'_2} = 0$ and $\overline{f'_1}$ is surjective. Put $T_1 = \ker \overline{f'_1}$. Then $\ker H(f) = (0, T_1, 0)$ by [6, Lemma 1. 4]. On the other hand, since $H(f)$ is irreducible, $H(f)$ is a proper epimorphism or proper monomorphism. It cannot be an epimorphism, so a monomorphism. It holds that $T_1 = 0$ and then $\overline{f'_1}$ is an isomorphism. Thus we have $M\Gamma \cong X_1$. Since $\mathrm{rank}_R\, M\Gamma = \mathrm{rank}_R\, (\ker g_1) = \mathrm{rank}_R\, X_1 + \mathrm{rank}_R X_2$, we conclude that $X_2 = 0$ and $\mathrm{Im}\, f_1 = \ker g_1$, so that $0 \to M\Gamma \to M'\Gamma \to M''\Gamma \to 0$ is exact. Hence by [6, Lemma 1. 4] $0 \to \Phi(M) \to \Phi(M') \to \Phi(M'') \to 0$ is exact.

Corollary 3. *Let $0 \to M \to M' \to M'' \to 0$ be an almost split sequence in* latt Λ *such that M is not a Γ-lattice. Then $0 \to \Phi(M) \to \Phi(M') \to \Phi(M'') \to 0$ is an almost split sequence in* $\mathrm{mod}_{\mathrm{sp}}\, C$.

Proof. By Proposition 2 $0 \to \Phi(M) \to \Phi(M') \to \Phi(M'') \to 0$ is exact and by Lemma 1 it doesn't split. Let $0 \to \Phi(M) \to \Phi(N) \to L \to 0$ be an almost split sequence starting from $\Phi(M)$ in $\mathrm{mod}_{\mathrm{sp}}\, C$. Then there exists a splitting epimorphism $\Phi(N) \to \Phi(M')$. We have a splitting epimorphism $N \to M'$ by Lemma 1. Applying the similar arguement to the irreducible map $M \to N$ and the almost split sequence $0 \to M \to M' \to M'' \to 0$ we have a splitting epimorphism $M' \to N$. Therefore, $M' \cong N$.

By Corollary 3 the remaining almost split sequences of latt Λ are those starting from Γ-lattices. Before investigating these almost split sequences we summarize the projective C-modules. An indecomposable projective C-module, up to isomorphism, is either a simple projective module of the form $(0, K_i, 0)\,(i=1, \dots, t)$ or $\Phi(M)$ for an indecomposable projective Λ-lattice M. For it holds that a C-module X is projective if and only if X is a projective object in $\mathrm{mod}_{\mathrm{sp}}\, C$, since C is a right peak ring. Thus statement follows from Theorem B.

In order to describe the behavior of the almost split sequences starting from Γ-lattices under the functor Φ, we prepare some well-known results about hereditary orders [4, Chapter 39]. Decompose $\Gamma = \Gamma_1 \oplus \dots \oplus \Gamma_k$ with each Γ_i indecomposable as a ring. Let $G_{ij}(j=1, \dots, a_i)$ be the representatives of the indecomposable projective Γ_i-lattices $(i=1, \dots, k)$. We number as follows for each $i(i=1, \dots, k)$ [4, (39.8)] ; $G_{ij} \cong G_{i,j+1}\,(\mathrm{rad}\, \Gamma_i)\,(j=i, \dots, a_i-1)$ and $G_{ia_i} \cong G_{i1}(\mathrm{rad}\, \Gamma_i)$. Put

$$\theta(G_{ij}) = \begin{cases} G_{i,j+1}, & \text{if } j=1, \dots, a_i-1 \\ G_{ij}, & \text{if } j=a_i. \end{cases}$$

Then $\theta(G_{ij})$ is the unique minimal over Γ-module of G_{ij} such that $\theta(G_{ij})/G_{ij}$ is a simple Γ-module. Of course, in the case that $j=a_i$ we identify G_{ia_i} to its isomorphic image in G_{i1}.

PROPOSITION 3. *Let G be an indecomposable Γ-lattice. Then the following holds.*

1) *If there exists an almost split sequence* $0 \to G \xrightarrow{f} L \xrightarrow{g} M \to 0$ *in* latt Λ, *then* $0 \to p \to \Phi(L) \to \Phi(M) \to 0$ *is an almost split sequence in* $\mathrm{mod}_{\mathrm{sp}}\, C$ *and* $p \cong (0, F, 0)$ *is a simple projective C-module where* $F = \mathrm{End}_B \overline{\theta(G)}$ *is a division ring.*

2) *If G is an injective Λ-lattice and f: $G \to N$ is an irreducible map in* latt Λ, *then there exists an irreducible map* $(0, F, 0) \to \Phi(N)$, *where F is the same as in* 1.

PROOF. 1) Since $\Phi(M)$ is not projective and $\Phi(g)$ is irreducible, the almost split sequence ending at $\Phi(M)$ in $\mathrm{mod}_{\mathrm{sp}}\, C$ is $0 \to p \to \Phi(L) \to \Phi(M) \to 0$. In order to prove $p \cong (0, F, 0)$ it suffices to show $(0, F, 0) \cong \ker \Phi(g)$. Let

g_1 be the same as in the proof of Lemma 1. Then g_1 is a splitting epimorphism and $L\Gamma = \ker g_1 \oplus Y$. Decompose $L\Gamma$ into a direct sum of indecomposable projective Γ-lattices, say $L\Gamma = L_1 \oplus \ldots \oplus L_m$. Then $LJ = L_1 J \oplus \ldots \oplus L_m J$ is also a direct sum of indecomposable Γ-lattices. Since $f(G)$ is a direct summand of LJ by Lemma 2, we can assume $f(G) = L_1 J$ and $\theta(G) = L_1$. It holds that $L\Gamma = \theta(G) \oplus X$ for a Γ-lattice X. Since $g_1 f(G) = 0$ and $\theta(G)/G$ is R-torsion, we conclude that $g_1(\theta(G)) = 0$. By rank arguement we have that $\ker g_1 \cap X = 0$. Then by an elementwise arguement $\ker g_1 \oplus Y = \theta(G) \oplus X$ deduces that $\ker g_1 = \theta(G)$, so that $\ker \bar{g}_1 = \overline{\theta(G)}$ for $\bar{g}_1 : \overline{L\Gamma} \to \overline{M\Gamma}$. On the other hand, since $\mathrm{Im}\, f \subset LJ$, $\bar{g}: \bar{L} \to \bar{M}$ is an isomorphism. It holds that $\ker H(g) \cong (0, \overline{\theta(G)}, 0)$. Applying the functor H' we have that $\ker \Phi(g) \cong (0, F, 0)$.
2) By Corollary 2 $\Phi(f) = 0$ and by Lemma 2 $\mathrm{Im}\, f$ is a direct summand of NJ. Thus the irreducible map f is a proper monomorphism such that coker f is a simple Λ-module by [10, Proposition 1.2], since G is an injective Λ-lattice. It holds that $\mathrm{Im}\, f = NJ$ and $NJ = \mathrm{rad}_\Lambda N$, so $N\Gamma$ is an indecomposable Γ-lattice. Thus we conclude that $N\Gamma \cong \theta(G)$, so that $\mathrm{rad}\, H(N) = \mathrm{rad}\, (\bar{N}, \overline{N\Gamma}) \cong (0, \overline{\theta(G)}, 0)$. Applying the functor H' it holds that $\mathrm{rad}\, \Phi(N) \cong (0, F, 0)$. By Lemma 2 and the paragraph succeeding Corollary 3, $\Phi(N)$ is projective. Hence there exists an irreducible map $(0, F, 0) \to \Phi(N)$.

COROLLARY 4. *If* $0 \to M \to M' \to M'' \to 0$ *is an exact sequence in* latt Λ *such that* $0 \to \Phi(M) \to \Phi(M') \to \Phi(M'') \to 0$ *is an almost tplit sequence in* $\mathrm{mod}_{\mathrm{sp}}$ C*. then it is an almost split sequence in* latt Λ.

PROOF. Keeping in mind that M'' is not a projective Λ-lattice this follows from Proposition 3 and Corollary 3.

PROOF OF THEOREM C. This follows directly from Lemma 3, Proposition 3 and Corollaries 3, 4.

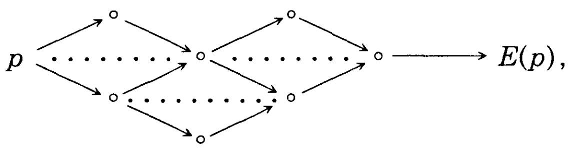
Following the notation provided before Proposition 3, we restate Theorem C more concretely. Put $p_{ij} = (0, \mathrm{End}_B \bar{G}_{ij}, 0)$ $(i = 1, \ldots, k;\ j = 1, \ldots, a_i)$. Then they are simple projective C-modules and each simple projective C-module is isomorphic to one of them. Note that $E(p_{ij})$, the injective hull of p_{ij}, is isomorphic to $\Phi(G_{ij})$ for all i, j. Let Q be the Auslander-Reiten quiver of $\mathrm{mod}_{\mathrm{sp}}$ C. Then Q contains each p_{ij} and $E(p_{ij})$ as its vertices. In Q, identify $E(p_{ij})$ with $p_{i,j+1}$ $(j = 1, \ldots, a_i - 1)$ and $E(p_{ia_i})$ with p_{i1} for all $i = 1, \ldots, k$, and then put the quiver obtained after this identification Q'. Then Q' is the Auslander-Reiten quiver of latt Λ, where the vertices identified by the above way correspond each other to indecomposable Γ-lattices. More precisely, $E(p_{ij})$, $p_{i,j+1}$ in Q correspond to G_{ij} in $Q'(j=1,\dots,a_i-1)$ and $E(p_{ia_i})$, p_{i1} in Q to G_{ia_i} in Q' for all i.

EXAMPLE. Let $\Lambda=\begin{pmatrix} R & R & R & R & R \\ r & R & r & R & R \\ r & r & R & R & r \\ r & r & r & R & r \\ r & r & r & r & R \end{pmatrix}$, $\Gamma_i=(R)_5$, $\Gamma_2=\begin{pmatrix} R & R\dots R \\ r & R\dots R \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ r & R\dots R \end{pmatrix}$

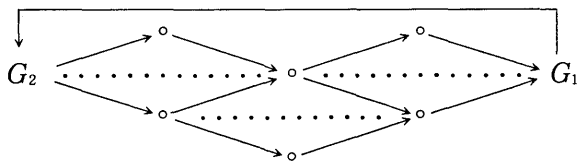
with $r=\pi R$. Then $\Lambda\subset\Gamma_1$ and $\Lambda\subset\Gamma_2$ satisfy our assumption. Firstly we consider the case of Γ_1. Since rad $\Gamma_1=(r)_5$, we have

$$C=\begin{pmatrix} k & k & k & k & k & k \\ 0 & k & 0 & k & k & k \\ 0 & 0 & k & k & 0 & k \\ 0 & 0 & 0 & k & 0 & k \\ 0 & 0 & 0 & 0 & k & k \\ 0 & 0 & 0 & 0 & 0 & k \end{pmatrix}$$

which is a path algebra of the bounden quiver $\begin{tikzcd}[sep=small] \circ \arrow[r] \arrow[d] & \circ \arrow[r] \arrow[d] & \circ \arrow[d] \\ \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$ with commuting cycles. The Auslander-Reiten quiver of $\mathrm{mod}_{\mathrm{sp}}\, C$ is the following ;



where p is simple projective and $E(p)$ its injective hull and the dotted lines indicate the τ-orbit of the Auslander-Reiten translation τ. Identifying p with $E(p)$ we get the Auslander-Reiten quiver of Λ ;



where G_2 is a Γ_1-lattice. For the case of Γ_2, C is a path algebra of the bounden quiver $\begin{tikzcd}[sep=small] & \circ \arrow[r] \arrow[d] & \circ \arrow[d] & \circ \arrow[r] & \circ \\ \circ \arrow[r] & \circ \arrow[r] & \circ & & \end{tikzcd}$ with a commuting cycle. The Auslander-Reiten quiver of $\mathrm{mod}_{\mathrm{sp}}\, C$ is the following ;

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & \circ \arrow[dr] & & \circ \arrow[dr] & & & \\
p_1 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \circ \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & E(p_1) & p_2 \arrow[r] & E(p_2), \\
 & \circ \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \circ \arrow[ur] & & & \\
 & & \circ \arrow[ur] & & & &
\end{tikzcd}
$$

where p_1 and $E(p_i)\,(i=1, 2)$ are the same as in the first case. Identifying p_1 with $E(p_2)$ and p_2 with $E(p_1)$ we get the same Auslander-Reiten quiver of latt Λ. In this case, $G_1 = (rRRRR)$ and $G_2 = (RRRRR)$ are Γ_2-lattices.

References

[1] Auslander, M., Reiten, I. : Representation theory of Artin algebras III, IV, Comm. Algebra, **3**(1975), 239–294, ibid., **5**(1977), 443–518.

[2] Green, E. L., Reiner, I. : Integral representations and diagrams, Michigan Math. J., **25**(1978), 53–84.

[3] Nishida, K. : Representations of orders and vector space categories, J. P. A. Algebra **33**(1984), 209–217.

[4] Reiner, I. : Maximal Orders (Academic Press, New York, 1975).

[5] Ringel, C. M., Roggenkamp, K. W. : Diagrammatic methods in the representation theories of orders, J. Algebra **60**(1979), 11–42.

[6] Roggenkamp, K. W. : Orders of global dimension two, Math. Z., **160**(1978), 63–67.

[7] Roggenkamp, K. W. : The lattice type of orders II, Integral Representations and applications (L. N. M. **882**, Springer, Berlin, 1981), 430–477.

[8] Roggenkamp, K. W. : Auslander-Reiten species for socle determined categories of hereditary algebras and for generalized Backstrom orders (Mitt. Math. Seminar, Giessen, **159,** 1983)

[9] Simson, D. : Vector space categories, right peak rings and their socle projective modules, J. Algebra, **92**(1985), 532–571.

[10] Schmidt, J. W. : Irreducible homomorphisms for lattices over orders, Bull. Austral. Math. Soc., **17**(1977), 109–124.

[11] Wiedemann, A. : Orders with loops in their Auslander-Reiten graph, Comm. Algebra, **9**(1981), 641–656.